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A C^*-algebraic formulation of local contextual hidden variables

Mićo Đurđević

Faculty of Physics, University of Belgrade, Studentski trg 12, PO Box 550, 11001 Beograd, Serbia, Yugoslavia

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Abstract. A general algebraic formalism for the study of local contextual hidden variable theories is presented. Contextuality, which is a property of all consistent hidden variable theories, is interpreted as a manifestation of the inadequacy of the algebra of quantum observables to completely describe physical systems. The Bell inequalities obstruction to locality is overcome by the use of a generalized probability theory.

1. Introduction

The aim of this paper is to investigate, in the framework of the algebraic approach, general properties of local hidden variable theories.

Under the notion of local hidden variable (LHV) theory we understand a theory based on the following concepts:

Individual system description. The theory describes individual physical systems. The description given by the theory is supposed to be finer than the quantum mechanical one.

Causality. In the framework of the theory a subquantum space Ω is defined. Elements of Ω (subquantum states) correspond to complete states of a given physical system. If a subquantum state $\omega \in \Omega$ of the system is known, then the outcome of any quantum measurement performed on it is determined.

Locality. The result of a quantum measurement S performed on the system in a given subquantum state $\omega \in \Omega$ does not depend on whether some other quantum measurement T, which does not interact with S, is performed jointly.

The lack-of-knowledge interpretation of quantum probabilities. The fact that quantum theory cannot give (except in some special cases) predictions for individual systems but only for statistical assemblies is interpreted as a consequence of the incompleteness of this theory. In accordance with this, quantum states (statistical operators in the standard quantum mechanical scheme) are understood as entities carrying information about a lack of knowledge of subquantum states. More precisely, each quantum state ρ becomes a 'probability measure' μ_ρ on Ω such that probabilities of quantum events in the state ρ are equal to probabilities, with respect to μ_ρ, of appropriate subquantum counterparts.

The principal question about the logical possibility of such a structure can be answered affirmatively. On the other hand, in constructing an LHV theory, we do not have absolute freedom: it is known that every LHV theory must satisfy specific requirements, related to obstructions given by the so-called 'no go' statements.

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Generally speaking, statements of this kind tell us that under certain external assumptions a hidden variable theory is not possible. Essentially, there are two kinds of 'no go' statements.

First of all, investigations of von Neumann [27], Gleason [19], Bell [4] and others (see [5]) show that the concept of a dispersion-free state (subquantum state) is incompatible with the structure of the lattice of quantum events (or with the algebra of quantum observables). From the logical viewpoint, these results are not at all related to the condition of subquantum locality nor to the ignorance interpretation of quantum states.

Problems of this kind can be overcome [2, 4, 5, 8, 9, 14, 16, 21, 35] by allowing the so-called *contextual* theories [32]. Their common and characteristic property is that the *value* of a given observable in a given subquantum state also depends on a measurement arrangement (context). As a rule, in contextual theories one and the same quantum observable, in the same subquantum state, possesses different values in different contexts.

The second obstruction comes from Bell's inequalities [3]. The violation of these inequalities by certain correlated two-particle quantum mechanical states implies that an LHV theory cannot reproduce the corresponding joint probabilities along the lines of classical (=Kolmogorovian) statistics. To put it another way, a hidden variable theory based on classical statistics and statistically compatible with quantum mechanics is necessarily (subquantally) non-local.

It is worth noting that, in accordance with the results of Summers and Werner [33, 34], essentially the same situation also holds in algebraic quantum field theory.

However, classical statistics is not the only way to describe lack-of-knowledge situations. Moreover, in the framework of suitable generalized statistics, a unification of quantum theory, locality and causality becomes possible [15, 17, 22, 23, 28, 30].

In summary, any local causal theory, statistically compatible with quantum mechanics, must be contextual and must be based on non-Kolmogorovian statistics.

Through this paper we shall deal with a generalized 'quantum structure', specified by a C^*-algebra Σ of 'quantum observables' together with a collection of allowed 'measurement contexts'. We shall distinguish two types of contexts: simple and composite. By definition, simple contexts correspond to single (one-particle) measurement situations, while composite contexts correspond to measurements composed of two or more mutually independent single measurements. We shall suppose that each simple context is realized as a commutative C^*-subalgebra of Σ, consisting of observables measurable in this context. Let $\mathbf{T}$ denote the family of such commutative C^*-subalgebras of Σ. On the other hand, each composite context is completely determined by simple contexts from which it is composed. In this sense we shall identify composite contexts with certain subsets of $\mathbf{T}$. Let $\hat{\mathbf{T}}$ be the family of all contexts. We shall assume that the following properties hold:

(i) The family $\mathbf{T}$ generates Σ. This is a condition of minimality for Σ, otherwise we could pass to a smaller algebra generated by $\mathbf{T}$.

(ii) The family $\hat{\mathbf{T}}$ is complete, in the sense that $A\in\hat{\mathbf{T}}$ and $B\subseteq A$ implies $B\in\hat{\mathbf{T}}$. This assumption is natural because of the above-mentioned interpretation of composite contexts.

(iii) For each $A=\{A_1,\ldots,A_n\}\in\hat{\mathbf{T}}$ the map $\varphi_A:A_1\otimes\ldots\otimes A_n\to\Sigma$ defined by $\varphi_A(\hat{a}_1\otimes\ldots\otimes\hat{a}_n)=\hat{a}_1\ldots\hat{a}_n$ is an injective *-homomorphism. In other words, simple contexts $A_1,\ldots,A_n$ forming a composite one are mutually compatible and uncorrelated.

It is clear that this abstract scheme includes, as a special case, all standard quantum-mechanical examples of distant measurements.

In this study, LHV theories are analysed in the framework of the contextual extensions approach [14]. The basic idea for this is an interpretation of contextuality as a manifestation of the inadequacy of the algebra Σ of quantum observables to completely describe the system. Following this idea we have to consider some finer algebra Σ' (of 'right' observables) equipped with a 'forgetting' epimorphism $\phi:\Sigma'\to\Sigma$, so that Σ can be obtained by factorizing Σ' through the ideal $\ker(\phi)$.

In section 2 we introduce a subclass of 'local contextual hidden variables (LCHV) extensions'. These extensions are relevant for the study of LHV theories. They will be characterized by four external assumptions.

The first one is 'locality', which leads to objects called local contextual (LC) extensions. After defining these entities we construct the 'maximal' LC extension. The theoretical relevance of this object lies in its universality, as well as in the fact that it satisfies another three assumptions, which complete the notion of LCHV extension and which are closely related to the ignorance interpretation of quantum states.

In section 2 we shall also introduce, for an arbitrary LCHV extension, the 'subquantum space' Ω and a notion of the value of a quantum observable $\hat{a}\in\Sigma$ in a subquantum state $\omega\in\Omega$ relative to a context $A\in\hat{\mathbf{T}}$.

The statistical foundation of quantum states, from the LCHV viewpoint, is a theme of section 3. Such a foundation becomes possible in the framework of 'contextual statistics', the base of which is a family $\mathfrak{P}$ of those subsets of the subquantum space Ω which are interpretable as 'quantally actualizable' subquantum events. The family $\mathfrak{P}$ is not Boolean σ-algebra. As we shall see, each quantum state ρ gives rise to a 'probability measure' $\mu_\rho:\mathfrak{P}\to[0,1]$ reproducing all quantum probabilities.

We then introduce, in a natural manner, a 'quantum interpreter' map

$$\vartheta: \{\text{quantally actualizable subquantum events}\}\to\{\text{quantum events}\}$$

which to a given subquantum event from $\mathfrak{P}$ attaches the corresponding quantum counterpart (a projector in the Hilbert space of quantum states). This map contains all the information about the measures μ_ρ.

We shall also analyse the possibility of extending ϑ from $\mathfrak{P}$ to more 'complete' domains. In particular, we shall show that, under some external assumptions, ϑ can be (uniquely) extended to an operator-valued map defined on the σ-class $\mathfrak{Q}$ (see [22, 23]) generated by $\mathfrak{P}$. If such extended ϑ is *positive* then each quantum state can be naturally realized as a probability measure on $\mathfrak{Q}$ (that is, μ_ρ are extendable to probability measures on $\mathfrak{Q}$). Only in this case is the statistical foundation of quantum states essentially the same as that proposed by Pitowsky and Gudder [22, 23, 28, 30].

However, in the general case, the extended ϑ is not positive. In other words, the 'contextual statistics' mentioned cannot always be incorporated in the approach of σ-classes.

Finally, we shall briefly discuss the 'classical case', when a quantum state is reducible to an ordinary (=Kolmogorovian) probability measure on Ω. Of course, due to Bell's inequalities, not all quantum states admit such reduction. We shall find a simple criterion characterizing states for which this is possible.

In section 4 some examples are considered, and a number of critical remarks are made.

We shall deal with unital C^*-algebras and unital homomorphisms. As a classical textbook on C^*-algebras, we refer to [13].

2. LCHV extensions

As already mentioned, to represent the 'quantum world' we take a pair $(\Sigma, \mathbf{T})$ consisting of a C^*-algebra Σ and a family $\mathbf{T}$ of commutative C^*-subalgebras of Σ (simple contexts), together with a collection $\hat{\mathbf{T}}$ (composite contexts) of subsets of $\mathbf{T}$, such that properties (i)–(iii) in section 1 hold.

For each $A=\{A_1,\ldots,A_n\}\in\hat{\mathbf{T}}$ we shall denote by $\mathrm{dom}(A)$ the commutative C^*-subalgebra of Σ generated by $A_1,\ldots,A_n$, by $i_A:\mathrm{dom}(A)\to\Sigma$ the inclusion map and by $X(A)$ the spectrum of $\mathrm{dom}(A)$. According to property (iii), the spectrum $X(A)$ of $\mathrm{dom}(A)$ is naturally homeomorphic to $X(A_1)\times\ldots\times X(A_n)$.

Now, we pass to the subquantum structure. Let us recall [14] that a contextual extension of $(\Sigma, \mathbf{T})$ is a triplet $(\Sigma', \phi, \{\iota_A;\ A\in\mathbf{T}\})$ where

(i) Σ' is a C^*-algebra;

(ii) $\phi:\Sigma'\to\Sigma$ is a $*$-homomorphism; (2.1)

(iii) ι_A: $A\to\Sigma'$ are $*$-homomorphisms such that $\phi\iota_A=i_A$;

(iv) Σ' is generated by subalgebras $\{\iota_A(A);\ A\in\mathbf{T}\}$.

In this paper we shall restrict ourselves to extensions, local in the sense of the following definition.

Definition 2.1. Contextual extension $(\Sigma', \phi, \{\iota_A;\ A\in\mathbf{T}\})$ is called *local*, iff

$$\iota_A(\hat{a})\iota_B(\hat{b})=\iota_B(\hat{b})\iota_A(\hat{a}) \tag{2.2}$$

for each $A, B\in\mathbf{T}$ such that $\{A, B\}\in\hat{\mathbf{T}}$, and each $\hat{a}\in A$ and $\hat{b}\in B$.

If $(\Sigma', \phi, \{\iota_A;\ A\in\mathbf{T}\})$ is a local contextual (LC) extension then for each $A=\{A_1,\ldots,A_n\}\in\hat{\mathbf{T}}$ the maps $\iota_{A_1},\ldots,\iota_{A_n}$ are uniquely extendable to a $*$-homomorphism ι_A: $\mathrm{dom}(A)\to\Sigma'$. It is easy to see that

$$\phi\iota_A=i_A \tag{2.3}$$

for each $A\in\hat{\mathbf{T}}$.

A trivial example of an LC extension is the triplet $(\Sigma, \mathrm{id}, \{i_A;\ A\in\mathbf{T}\})$. This is the 'minimal' extension. We pass on to the construction of the 'maximal' extension.

Let $Lc(\Sigma, \mathbf{T})$ be the C^*-algebra generated by the set of elements $\{(\hat{a}, A);\ A\in\mathbf{T}, \hat{a}\in A\}$ and the following relations:

$$\begin{aligned}
&(1, A)=1\\
&(\alpha\hat{a}+\beta\hat{b}, A)=\alpha(\hat{a}, A)+\beta(\hat{b}, A) \qquad \alpha,\beta\in C\\
&(\hat{a}, A)(\hat{b}, A)=(\hat{a}\hat{b}, A)\\
&(\hat{a}, A)^*=(\hat{a}^*, A)\\
&(\hat{a}, A)(\hat{b}, B)=(\hat{b}, B)(\hat{a}, A) \qquad \text{if } \{A, B\}\in\hat{\mathbf{T}}.
\end{aligned} \tag{2.4}$$

For each $A\in\hat{\mathbf{T}}$ we can define a $*$-homomorphism $\hat{\iota}_A$: $\mathrm{dom}(A)\to Lc(\Sigma, \mathbf{T})$ by the formula

$$\hat{\iota}_A(\hat{a}_1\ldots\hat{a}_n)=(\hat{a}_1, A_1)\ldots(\hat{a}_n, A_n) \tag{2.5}$$

where $A=\{A_1,\ldots,A_n\}$. The algebra $Lc(\Sigma, \mathbf{T})$ possesses the following *universal* property.

Proposition 2.1. Let Σ' be a C^*-algebra and $\{\lambda_A : A \to \Sigma'; A \in \mathbf{T}\}$ a family of homomorphisms such that

$$\lambda_A(\hat{a})\lambda_B(\hat{b}) = \lambda_B(\hat{b})\lambda_A(\hat{a}) \tag{2.6}$$

whenever $\{A, B\} \in \hat{\mathbf{T}}$.

(i) There exists the unique *-homomorphism $\lambda : Lc(\Sigma, \mathbf{T}) \to \Sigma'$ such that $\lambda\hat{\iota}_A = \lambda_A$ for each $A \in \mathbf{T}$.

(ii) If subalgebras $\{\lambda_A(A); A \in \mathbf{T}\}$ generate Σ' then λ is surjective.

Proof. The existence and the uniqueness of λ directly follow from the definition of $Lc(\Sigma, \mathbf{T})$. As a *-homomorphism between C^*-algebras, λ has the closed range $R(\lambda)$. If $\{\lambda_A(A) : A \in \mathbf{T}\}$ generate Σ' then $R(\lambda)$ is dense in Σ', which is possible only if $R(\lambda) = \Sigma'$. □

Let us apply this proposition to the following situation: $\Sigma' = \Sigma$, $\lambda_A = i_A$. We conclude that there exists the unique *-homomorphism $\hat{\phi} : Lc(\Sigma, \mathbf{T}) \to \Sigma$ such that $\hat{\phi}\hat{\iota}_A = i_A$, for each $A \in \mathbf{T}$. (Hence, $\hat{\phi}\hat{\iota}_A = i_A$ holds for each $A \in \hat{\mathbf{T}}$.)

Proposition 2.2. (i) The triplet $(Lc(\Sigma, \mathbf{T}), \hat{\phi}, \{\hat{\iota}_A; A \in \mathbf{T}\})$ is an LC extension of $(\Sigma, \mathbf{T})$.

(ii) If $(\Sigma', \phi, \{\iota_A; A \in \mathbf{T}\})$ is an LC extension, then there exists the unique *-homomorphism $\lambda : Lc(\Sigma, \mathbf{T}) \to \Sigma'$ such that

$$\iota_A = \lambda\hat{\iota}_A \qquad \text{for each } A \in \hat{\mathbf{T}}. \tag{2.7}$$

Proof. Property (i) follows directly from definition 2.1. Let $(\Sigma', \phi, \{\iota_A; A \in \mathbf{A}\})$ be an arbitrary LC extension. According to proposition 2.1, there exists the unique *-homomorphism $\lambda : Lc(\Sigma, \mathbf{T}) \to \Sigma'$ such that (2.7) holds for each $A \in \mathbf{T}$. If $A = \{A_1, \ldots, A_n\} \in \hat{\mathbf{T}}$ then $\iota_A(\hat{a}_1 \ldots \hat{a}_n) = \iota_{A_1}(\hat{a}_1) \ldots \iota_{A_n}(\hat{a}_n) = \lambda\hat{\iota}_{A_1}(\hat{a}_1) \ldots \lambda\hat{\iota}_{A_n}(\hat{a}_n) = \lambda\hat{\iota}_A(\hat{a}_1 \ldots \hat{a}_n)$, for each $\hat{a}_1 \in A_1, \ldots, \hat{a}_n \in A_n$. By the use of linearity and continuity we conclude that (2.7) holds. □

We shall now introduce three conditions which will ensure the lack-of-knowledge interpretation of quantum states. From this moment, we shall assume that Σ is faithfully represented in a Hilbert space H of 'quantum states'. Let $(\Sigma', \phi, \{\iota_A; A \in \mathbf{T}\})$ be an LC extension.

Property (a). Let $\mathrm{com}(\Sigma')$ be the ideal in Σ' generated by commutators. Then, $1 \notin \mathrm{com}(\Sigma')$. Equivalently, the set of characters (non-trivial multiplicative linear (necessarily Hermitian, or *) functionals) of Σ' is non-void.

Let us suppose that (a) holds and denote by Ω the set of characters of Σ'. Endowed with a *-weak topology, Ω is compact. The map $\pi : \Sigma' \to C(\Omega)$ defined by $\pi(a')(\omega) = \omega(a')$, where $C(\Omega)$ is the C^*-algebra of complex continuous functions on Ω, is surjective and $\ker(\pi) = \mathrm{com}(\Sigma')$. Consequently, $C(\Omega)$ is naturally isomorphic to $\Sigma'/\mathrm{com}(\Sigma')$.

The following lemma gives a characterization of elements of Ω.

Lemma 2.3. A functional $\omega : \Sigma' \to C$ belongs to Ω if and only if:

(i) It is a state;

(ii) Its dispersion on each element $\iota_A(\hat{a})$, where $\hat{a} = \hat{a}^+ \in A$ and $A \in \mathbf{T}$, is equal to zero.

Proof. It is easy to see that each character of Σ' satisfies (i) and (ii). Let us suppose that $\omega:\Sigma'\to C$ satisfies (i) and (ii). For each $b\in\Sigma'$, $A\in\mathbf{T}$ and $\hat{a}=\hat{a}^+\in A$ one then has

$$|\omega(b[\iota_A(\hat{a})-\omega(\iota_A(\hat{a}))1])|^2\leq\omega(bb^*)\omega[(\iota_A(\hat{a})-\omega(\iota_A(\hat{a}))1)^2]=0.$$

In other words,

$$\omega(b\iota_A(\hat{a}))=\omega(b)\omega(\iota_A(\hat{a})). \tag{2.8}$$

From this we conclude that

$$\omega(ba')=\omega(b)\omega(a') \tag{2.9}$$

holds for each $b\in\Sigma'$ and

$a'\in$ {the *-algebra generated by elements $\iota_A(\hat{a})$, where $A\in\mathbf{T}$ and $\hat{a}\in A$.}

Finally, by the use of continuity of ω, and property (2.1; iv) we conclude that (2.9) holds for each $b\in\Sigma'$ and $a'\in\Sigma'$. Thus $\omega\in\Omega$. □

We can think of characters $\omega\in\Omega$ as of 'subquantum states'. For each $A\in\hat{\mathbf{T}}$, $\hat{a}\in\mathrm{dom}(A)$ and $\omega\in\Omega$ the number $\omega(\iota_A(\hat{a}))$ can be interpreted as the *value* of $\hat{a}$ in the subquantum state ω, relative to the context A.

We pass to the second condition. For each $A\in\hat{\mathbf{T}}$ let F_A: $\mathrm{dom}(A)\to C(\Omega)$ be a *-homomorphism defined by $F_A=\pi\iota_A$. Let us denote by $l(\Omega)$ the lineal in $C(\Omega)$ generated by subalgebras $F_A(\mathrm{dom}(A))$.

Property (b): There exists a linear map $j:l(\Omega)\to\Sigma'$ such that

$$jF_A=\iota_A \qquad \text{for each } A\in\hat{\mathbf{T}}. \tag{2.10}$$

Let us suppose that (b) holds.

Lemma 2.4. The map j is Hermitian and determined uniquely by (2.10). For each $f\in l(\Omega)$ one has $\pi j(f)=f$.

Proof. This is a direct consequence of definitions of j, F_A's and the space $l(\Omega)$. □

Lemma 2.5. For each $A\in\hat{\mathbf{T}}$ the map F_A: $\mathrm{dom}(A)\to C(\Omega)$ is injective.

Proof. According to property (2.1; iii) one has $\phi jF_A(\hat{a})=\hat{a}$ for each $A\in\hat{\mathbf{T}}$ and $\hat{a}\in\mathrm{dom}(A)$, which implies the injectivity. □

This lemma can be reformulated as follows: for each $A\in\hat{\mathbf{T}}$ let $\pi_A:\Omega\to X(A)$ be a map defined by

$$\pi_A(\omega)=\omega\iota_A. \tag{2.11}$$

This map is continuous, according to definition of topology in Ω. Lemma 2.5 is equivalent to surjectivity of all π_A's.

For each finite set $F\subseteq\mathbf{T}$, let $l_F(\Omega)\subseteq l(\Omega)$ be the sublineal in $l(\Omega)$ generated by subalgebras $\{F_A(\mathrm{dom}(A)); A\subseteq F\}$. Our last condition will be the following.

Property (c). For each finite set $F\subseteq\mathbf{T}$, the composition $\phi j_F:l_F(\Omega)\to\Sigma$ is continuous. Here, $j_F=(j|l_F(\Omega))$.

The following lemma gives a simple sufficient condition for property (c).

Lemma 2.6. Let us suppose that $(\Sigma', \phi, \{\iota_A; A \in \mathbf{T}\})$ satisfies properties (a) and (b). If the spaces $l_F(\Omega)$ and $jl_F(\Omega)$ are closed, then the maps $j_F : l_F(\Omega) \to \Sigma'$ are continuous and, consequently, property (c) holds.

Proof. If $l_F(\Omega)$ is closed in $C(\Omega)$ then, according to the continuity of π, the space $\pi^{-1}(l_F(\Omega))$ is closed in Σ'. Because of lemma 2.4, the formula

$$f(a') = \pi(a') \oplus (a' - j\pi(a'))$$

defines a map $f: \pi^{-1}(l_F(\Omega)) \to l_F(\Omega) \oplus \ker \pi$. This map is bijective and continuous. According to the closed graphic theorem, f^{-1} is continuous. In particular, the restriction $(f^{-1}|l_F(\Omega)) = j_F$ is continuous. □

Definition 2.2. An LC extension $(\Sigma', \phi, \{\iota_A; A \in \mathbf{T}\})$ is called *local contextual hidden variables* (LCHV) extension of $(\Sigma, \mathbf{T})$ iff it satisfies properties (a), (b) and (c).

There exists at least one LCHV extension.

Proposition 2.7. The triplet $(Lc(\Sigma, \mathbf{T}), \hat{\phi}, \{\hat{\iota}_A; A \in \mathbf{T}\})$ is an LCHV extension.

Proof. Let us first observe that the space of characters of $Lc(\Sigma, \mathbf{T})$ can be naturally identified with the direct product

$$\Pi = \prod_{A \in \mathbf{T}} X(A).$$

Indeed, if we choose for each $A \in \mathbf{T}$ a character $\varphi_A \in X(A)$ then proposition 2.1 ensures the existence of the unique character $\varphi: Lc(\Sigma, \mathbf{T}) \to C$ which is completely determined by compositions $\varphi\hat{\iota}_A \in X(A)$.

Let us fix $\omega_0 \in \Pi$. For a given $A = \{A_1, \ldots, A_n\} \in \hat{\mathbf{T}}$ we define $L_A \subseteq C(\Pi)$ to be the lineal consisting of all functions ψ of the form

$$\psi(\omega) = \pi_A(\omega)(f) \tag{2.12}$$

where $\pi_A = \pi_{A_1} \times \ldots \times \pi_{A_n} : \Pi \to X(A) \simeq X(A_1) \times \ldots \times X(A_n)$ is a natural projection, $f \in \mathrm{dom}(A)$ and $f((\ ), \ldots, \pi_{A_i}(\omega_0), \ldots, (\ )) = 0$ for each $1 \leqslant i \leqslant \eta$. It is easy to see that lineals $\{L_A; A \in \hat{\mathbf{T}}\}$ are mutually linearly independent and that, for each finite set $F \subseteq \mathbf{T}$ the sum $l_F(\Pi) = \Sigma^{(+)}_{A \subseteq F} L_A$ is closed in $C(\Pi)$. The formula $\hat{j}(\psi) = \hat{\iota}_A(f)$, where $\psi \in L_A$, consistently defines the map $\hat{j}: l(\Pi) \to Lc(\Sigma, \mathbf{T})$ which satisfies $\hat{j}\hat{F}_A = \hat{\iota}_A$ for each $A \in \mathbf{T}$. Here, $\hat{F}_A = \hat{\pi}\hat{\iota}_A$, $l(\Pi) = \Sigma^{(+)}_{A \in \hat{\mathbf{T}}} L_A$ and $\hat{\pi}: Lc(\Sigma, \mathbf{T}) \to C(\Pi)$ is the factor projection. It is easy to see that spaces $\hat{j}l_F(\Pi)$ are closed. According to lemma 2.6, $\hat{j}_F$ are continuous and $(Lc(\Sigma, \mathbf{T}), \hat{\phi}, \{\hat{\iota}_A; A \in \mathbf{T}\})$ is an LCHV extension. □

3. The statistical foundation of quantum states

Through this section we shall deal with an arbitrary LCHV extension $(\Sigma', \phi, \{\iota_A; A \in \mathbf{T}\})$. By definition, states ρ on Σ that are of the form $\rho(\hat{a}) = \mathrm{Tr}(\hat{\rho}\hat{a})$, where $\hat{\rho}$ is a statistical operator in H, will be called 'quantum states'. For each quantum state ρ and a finite set $F \subseteq \mathbf{T}$ the map $\rho\phi j_F : l_F(\Omega) \to C$ is Hermitian and continuous. Let

$$d_{\rho,F} = |\rho\phi j_F| \tag{3.1}$$

be the norm of this functional. Evidently, $1 \leqslant d_{\rho,F} \leqslant |\phi j_F|$.

Lemma 3.1. For each quantum state ρ and finite set $F \subseteq \mathbf{T}$ there exists a Hermitian linear functional $\rho'_F : C(\Omega) \to C$ such that

$$\text{(i)} \qquad \rho(\hat{a}) = \rho'_F(F_A(\hat{a})) \tag{3.2}$$

for each context $A \subseteq F$ and $\hat{a} \in \mathrm{dom}(A)$;

(ii) one has

$$|\rho'_F(f)| \leq d_{\rho,F} \max_{\omega \in \Omega} \{|f(\omega)|\} \qquad \text{for each } f \in C(\Omega). \tag{3.3}$$

Proof. For each quantum state ρ, a finite set $F \subseteq \mathbf{T}$ and context $A \subseteq F$ one has

$$\rho\phi j_F F_A = \rho\phi\iota_A = \rho i_A. \tag{3.4}$$

Now, according to the Hahn–Banach theorem, $\rho\phi j_F$ can be extended, without changing its norm, to a hermitian continuous functional $\rho'_F : C(\Omega) \to C$. □

For a given $A \in \hat{\mathbf{T}}$, let us consider a Boolean σ-algebra $\mathfrak{P}_A$ of sets of the form $\pi_A^{-1}(\Lambda)$, where Λ is an arbitrary Baire set in $X(A)$.

Speaking 'subquantally', the family $\mathfrak{P}_A$ consists precisely of those subquantum events that are actualizable in the context A.

The union $\mathfrak{P} = \bigcup_{A \in \hat{\mathbf{T}}} \mathfrak{P}_A$ then consists of subquantum events which are interpretable quantally.

Now we are ready to show that each quantum state ρ can be realized as a 'probability measure' μ_ρ on $(\Omega, \mathfrak{P})$. Let us denote by $\mu_{\rho,A}$ the probability measure on (the Baire field of) $X(A)$ induced, via the Riesz theorem, by a state $\rho i_A : \mathrm{dom}(A) \to C$.

Proposition 3.2. For every quantum state ρ there exists the unique map $\mu_\rho : \mathfrak{P} \to [0, 1]$ such that

$$\mu_\rho(\Lambda) = \mu_{\rho,A}(\pi_A(\Lambda)) \qquad \text{for each } A \in \hat{\mathbf{T}} \text{ and } \Lambda \in \mathfrak{P}_A. \tag{3.5}$$

Proof. It is clear that the measure μ_ρ is, if it exists, unique. Let us show its existence. For each finite set $F \subseteq \mathbf{T}$, the functional ρ'_F introduced in lemma 3.1 naturally induces a real-valued measure ν_F on the Baire σ-field $B(\Omega)$. This measure satisfies

$$\nu_F(\Lambda) = \mu_{\rho,A}(\pi_A(\Lambda)) \qquad \text{for each context } A \subseteq F \text{ and } \Lambda \in \mathfrak{P}_A. \tag{3.6}$$

In particular, if $\Lambda \in \mathfrak{P}_{A_1} \cap \mathfrak{P}_{A_2}$ and F is chosen such that $A_1, A_2 \subseteq F$ then $\mu_{\rho,A_1}(\pi_{A_1}(\Lambda)) = \mu_{\rho,A_2}(\pi_{A_2}(\Lambda))$.

Therefore, for $\Lambda \in \mathfrak{P}_A$, the formula $\mu_\rho(\Lambda) = \mu_{\rho,A}(\pi_A(\Lambda))$ consistently defines a map with the desired properties. □

We can think of $\mu_\rho : \mathfrak{P} \to [0, 1]$ as the 'probability measure' on $(\Omega, \mathfrak{P})$ measuring the lack of knowledge, inherent in ρ, about subquantum states $\omega \in \Omega$.

Proposition 3.2 is not in contradiction with Bell's inequalities: for a state ρ violating these inequalities, we can only conclude that a classical probability measure $\mu : B(\Omega) \to [0, 1]$, which extends the map μ_ρ, does not exist.

The fact that μ_ρ is not, in general, extendable to a probability measure on $B(\Omega)$ does not deny the possibility of the ignorance interpretation of ρ.

Indeed, there is no physical sense in attaching a probability to subquantum event $\Lambda \in B(\Omega)\backslash\mathfrak{P}$, because such events are not accessible through any quantum measurement.

According to the above-mentioned interpretation of elements of the family $\mathfrak{P}$ as subquantum events possessing a quantum meaning, there should exist the corresponding 'quantum interpreter' map:

$$\{\text{elements of } \mathfrak{P}\} \to \{\text{quantum events}\}.$$

We shall now prove the existence and the uniqueness of such an entity, and investigate its main properties.

Let us denote by $\tilde{\mathfrak{P}}$ the minimal family of (Baire) sets in Ω which contains $\mathfrak{P}$ such that:

(i) If $\Lambda \in \tilde{\mathfrak{P}}$ then

$$\Omega/\Lambda \in \tilde{\mathfrak{P}} \tag{3.7}$$

(ii) If Ω is decomposed into a disjoint union

$$\Omega = \bigcup_{i \in N} \Lambda_i$$

and if $\Lambda_i \in \tilde{\mathfrak{P}}$ for each $i \in N$, then

$$\left(\bigcup_{i \in S} \Lambda_i\right) \in \tilde{\mathfrak{P}} \qquad \text{for each } S \subseteq N. \tag{3.8}$$

Let us denote by $\mathfrak{Q}$ the minimal σ-class of (Baire) sets in Ω which contains $\mathfrak{P}$. We recall [22, 23] that the σ-class is a family of sets closed under complementation and countable disjoint unions, and which contains $\varnothing$.

Similarly, for each finite set $F \subseteq \mathbf{T}$ let $\mathfrak{P}_F$ be the union of those $\mathfrak{P}_A$ for which $A \subseteq F$, let $\mathfrak{Q}_F \subseteq \mathfrak{Q}$ be the σ-class generated by $\mathfrak{P}_F$ and let $\tilde{\mathfrak{P}}_F \subseteq \tilde{\mathfrak{P}}$ be the minimal family of (Baire) sets in Ω which contains $\mathfrak{P}_F$ and satisfies (i) and (ii) above. It is easy to see that $\tilde{\mathfrak{P}}_F \subseteq \mathfrak{Q}_F$ and $\tilde{\mathfrak{P}} \subseteq \mathfrak{Q}$. If $F_1 \subseteq F_2$ then $\mathfrak{P}_{F_1} \subseteq \mathfrak{P}_{F_2}$, $\tilde{\mathfrak{P}}_{F_1} \subseteq \tilde{\mathfrak{P}}_{F_2}$ and $\mathfrak{Q}_{F_1} \subseteq \mathfrak{Q}_{F_2}$.

Let $P(H)$ be the projector lattice in H.

Proposition 3.3. (i) There exists one and only one projector-valued map $\vartheta : \mathfrak{P} \to P(H)$ such that

$$\mathrm{Tr}(\hat{\rho}\vartheta(\Lambda)) = \mu_\rho(\Lambda) \tag{3.9}$$

for each $\Lambda \in \mathfrak{P}$ and each statistical operator $\hat{\rho}$ in H;

(ii) For each $A \in \hat{\mathbf{T}}$ and $\Lambda \in \mathfrak{P}_A$ one has

$$\vartheta(\Lambda) = c_A[\pi_A(\Lambda)] \tag{3.10}$$

where $c_A : B(X(A)) \to P(H)$ is the spectral measure associated with the inclusion $i_A : \mathrm{dom}(A) \to L(H)$;

(iii) For each finite set $F \subseteq \mathbf{T}$, the map $\vartheta_F = (\vartheta|\mathfrak{P}_F)$ can be uniquely extended to the Hermitian operator-valued ultraweakly σ-additive map $\tilde{\vartheta}_F : \mathfrak{Q}_F \to L(H)$;

(iv) The images of the maps $\tilde{\vartheta}_F$ are contained in the bicommutant of Σ in $L(H)$;

(v) The restrictions of $\tilde{\vartheta}_F$ on $\tilde{\mathfrak{P}}_F$ are projector-valued;

(vi) If $F_1 \subseteq F_2$ then $(\tilde{\vartheta}_{F_2}|\mathfrak{Q}_{F_1}) = \tilde{\vartheta}_{F_1}$.

Proof. For $\Lambda \in \mathfrak{P}_A$ we define $\vartheta(\Lambda)$ to be equal to $c_A[\pi_A(\Lambda)]$. A necessary and sufficient condition for the existence of a 'global' map $\vartheta : \mathfrak{P} \to P(H)$ is that $c_A[\pi_A(\Lambda)]$ does not depend on A.

According to the same reasoning as in the previous proof, there exists, for each statistical operator $\hat{\rho}$ in H and each finite set $F \subseteq \mathsf{T}$, a real-valued measure ν_F on $B(\Omega)$ such that

$$\nu_F(\Lambda) = \mathrm{Tr}[\hat{\rho} c_A(\pi_A(\Lambda))] \tag{3.11}$$

for each $A \subseteq F$ and $\Lambda \in \mathfrak{P}_A$. Especially, if $\Lambda \in \mathfrak{P}_{A_1} \cap \mathfrak{P}_{A_2}$ and $A_1 \cup A_2 \subseteq F$, then

$$\mathrm{Tr}[\hat{\rho} c_{A_1}(\pi_{A_1}(\Lambda))] = \mathrm{Tr}[\hat{\rho} c_{A_2}(\pi_{A_2}(\Lambda))] \tag{3.12}$$

for each $\hat{\rho}$, which implies that $c_{A_1}(\pi_{A_1}(\Lambda)) = c_{A_2}(\pi_{A_2}(\Lambda))$.

If another map $\vartheta : \mathfrak{P} \to P(H)$ satisfying (3.9) is given, then $\mathrm{Tr}(\hat{\rho}(\vartheta(\Lambda) - \vartheta'(\Lambda))) = 0$, for each $\hat{\rho}$. This implies $\vartheta(\Lambda) = \vartheta'(\Lambda)$.

To show (iii), let us observe that for each Hermitian operator $\hat{f} \in L^1(H)$ of the trace-class and finite set $F \subseteq \mathsf{T}$ there exists a real-valued measure φ_F on $B(\Omega)$ such that

$$\varphi_F(\Lambda) = \mathrm{Tr}(\hat{f}\vartheta(\Lambda)) \tag{3.13}$$

for each $\Lambda \in \mathfrak{P}_F$. This is a direct consequence of (3.10) and (3.11) and the fact that $\hat{f}$ is a difference of positive operators from $L^1(H)$.

Let us denote by $\mathfrak{Q}(\hat{f}, F)$ the family of Baire sets $\Lambda \subseteq \Omega$ for which $\varphi_F(\Lambda)$ does not depend on the choice of φ_F, for a given $\hat{f}$ and F. clearly, $\mathfrak{Q}(\hat{f}, F)$ is a σ-class. According to (3.13) one has $\mathfrak{P}_F \subseteq \mathfrak{Q}(\hat{f}, F)$ and therefore $\mathfrak{Q}_F \subseteq \mathfrak{Q}(\hat{f}, F)$.

Consequently, for each $\Lambda \in \mathfrak{Q}_F$, the formula

$$\psi_{\Lambda,F}(\hat{f}) = \varphi_F(\Lambda) \tag{3.14}$$

determines a real-valued functional $\psi_{\Lambda,F}$ on the Hermitian part of $L^1(H)$. It is easy to see that

$$|\psi_{\Lambda,F}(\hat{f})| \leqslant |\phi j_F| \, \mathrm{Tr}[|f|] \tag{3.15}$$

in other words, $\psi_{\Lambda,F}$ is continuous. Consequently, there exists the unique Hermitian operator $\tilde{\vartheta}_F(\Lambda)$ on H such that

$$\psi_{\Lambda,F}(\hat{f}) = \mathrm{Tr}(\tilde{\vartheta}_F(\Lambda)\hat{f}) \qquad \text{for each Hermitian } \hat{f} \in L^1(H). \tag{3.16}$$

The map $\tilde{\vartheta}_F : \mathfrak{Q}_F \to L(H)$ extends ϑ_F. Let us suppose that $\Lambda = \bigcup_{i \in N} \Lambda_i$ where Λ_i are mutually disjoint. Then for each Hermitian $\hat{f} \in L^1(H)$ one has

$$\begin{aligned}\mathrm{Tr}[\tilde{\vartheta}_F(\Lambda)\hat{f}] &= \psi_{\Lambda,F}(\hat{f}) = \varphi_F(\Lambda) = \sum_{i \in N} \varphi_F(\Lambda_i) = \sum_{i \in N} \psi_{\Lambda_i,F}(\hat{f}) \\ &= \sum_{i \in N} \mathrm{Tr}[\tilde{\vartheta}_F(\Lambda_i)\hat{f}].\end{aligned}$$

This shows that $\tilde{\vartheta}_F$ is an ultraweakly σ-additive map. It is easy to see that $\tilde{\vartheta}_F$ is unique, as an ultraweakly σ-additive extension of ϑ_F.

Let us now denote by $\mathfrak{P}'_F$ the subfamily of $\mathfrak{Q}_F$ consisting of all Λ such that $\tilde{\vartheta}_F(\Lambda) \in P(H)$. Clearly, $\mathfrak{P}_F \subseteq \mathfrak{P}'_F$ and $\mathfrak{P}'_F$ satisfy properties (3.7) and (3.8) because if a sum of projectors is equal to $\hat{I}$, then they are mutually orthogonal. According to the definition of $\tilde{\mathfrak{P}}_F$ one has $\tilde{\mathfrak{P}}_F \subseteq \mathfrak{P}'_F$.

To prove (iv) let us consider the family $\mathfrak{Q}'_F$ of all $\Lambda \in \mathfrak{Q}_F$ which satisfy $\tilde{\vartheta}_F(\Lambda) \in (\Sigma)''$. Owing to the closeness of $(\Sigma)''$ in the ultraweak operator topology, $\mathfrak{Q}'_F$ is a σ-class. Evidently, $\mathfrak{P}_F \subseteq \mathfrak{Q}'_F$. Consequently, $\mathfrak{Q}_F = \mathfrak{Q}'_F$.

Finally, property (vi) directly follows from the uniqueness of maps $\{\tilde{\vartheta}_F;\, F \subseteq \mathbf{T}\}$. □

The map ϑ is, in general, not extendable on the 'global' σ-class $\mathfrak{Q}$, generated by the whole $\mathfrak{P}$. However, if a collection of numbers $\{|\phi j_F|;\, F \subseteq \mathbf{T}\}$ has an upper bound (this is equivalent to the continuity of ϕj) then essentially the reasoning as in the above proof shows that ϑ admits the unique Hermitian ultraweak σ-additive extension $\tilde{\vartheta}: \mathfrak{Q} \to \Sigma''$ and that $\tilde{\vartheta}(\tilde{\mathfrak{P}}) \subseteq P(H)$.

To simplify discussion, let us suppose for a moment that ϕj is continuous. Then the formula

$$\tilde{\mu}_\rho(\Lambda) = \mathrm{Tr}(\hat{\rho}\tilde{\vartheta}(\Lambda)) \tag{3.17}$$

determines a real-valued measure on $(\Omega, \mathfrak{Q})$ which extends μ_ρ. It is easy to see that $\tilde{\mu}_\rho(\tilde{\mathfrak{P}}) \subseteq [0, 1]$ for each quantum state ρ. Hence the restriction $\tilde{\mu}_\rho|\tilde{\mathfrak{P}}$ is interpretable as a 'probability measure'. (Because of the σ-additivity, the restrictions of $\tilde{\mu}_\rho$ on Boolean sub-σ-algebras of $\tilde{\mathfrak{P}}$ are ordinary probability measures.) However, for an arbitrary $\Lambda \in \mathfrak{Q}$ it may happen that $\tilde{\mu}_\rho(\Lambda) < 0$ or $\tilde{\mu}_\rho(\Lambda) > 1$. In other words, contextual statistics (defined on $(\Omega, \mathfrak{P})$) is not, in general, extendable to a statistics on $(\Omega, \mathfrak{Q})$. Of course, if $\tilde{\vartheta}$ is *positive*, then μ_ρ are probability measures on $(\Omega, \mathfrak{Q})$.

At the end of this section, we give a criterion for the possibility of reducing a quantum state ρ to an ordinary probability measure on Ω.

Proposition 3.4. The following conditions are equivalent:

(i) $d_{\rho,F} = 1$, for each finite set $F \subseteq \mathbf{T}$,

(ii) There exists a probability measure $\mu: B(\Omega) \to [0, 1]$ such that

$$\rho(\hat{a}) = \int_\Omega F_A(\hat{a})(\omega)\, \mathrm{d}\mu(\omega) \qquad \text{for each } A \in \hat{\mathbf{T}} \text{ and } \hat{a} \in \mathrm{dom}(A). \tag{3.18}$$

Proof. If (i) holds then for each finite $F \subseteq \mathbf{T}$ there exists a Hermitian linear functional ρ'_F on $C(\Omega)$, of the unit norm, satisfying (3.2). On the other hand, it is well known that $|\rho'_F| = \rho'_F(1)$ implies the positivity of ρ'_F. Consequently, each ρ'_F is a state on $C(\Omega)$. Due to the compactness of the set of states in the *-weak topology of $C(\Omega)$, there exists a subnet of the net $F \to \rho'_F$ which converges (in the *-weak topology) to a state ρ'. It is easy to see that

$$\rho'(F_A(\hat{a})) = \rho(\hat{a}) \qquad \text{for each } A \in \hat{\mathbf{T}} \text{ and } \hat{a} \in \mathrm{dom}(A). \tag{3.19}$$

Let us denote by μ a probability measure on $B(\Omega)$ which, via the Riesz theorem, corresponds to ρ'. Equation (3.19) can then be rewritten in the form (3.18).

Conversely, if (ii) holds, then the positive functional

$$C(\Omega) \ni f \to \int_\Omega f(\omega)\, \mathrm{d}\mu(\omega) \in C$$

extends the maps $\rho\phi j_F$, which implies $d_{\rho,F} = 1$. □

4. Examples and critical remarks

(i) The generalized concept of probability, presented in this paper, requires an additional justification: every 'probability measure' $\mu_\rho : \mathfrak{P} \to [0, 1]$ should be interpretable in terms of relative frequencies of occurrence. The possibility of such an interpretation is proven in [17]. Informally speaking, appropriate contextual statistics on the spaces $\{\Omega^k; k \in N\}$ of sequences ensures the possibility for the relative-frequencies interpretation of probabilities. (The same situation holds in classical statistics, where the same statistics on spaces of sequences ensures the relative-frequencies interpretation of probabilities.)

However, in contrast to the classical case, probabilities of events from a given finite family $F \subseteq \mathfrak{P}$ are (unless $F \subseteq \mathfrak{P}_A$ for some $A \in \hat{\mathrm{T}}$) generally not interpretable in terms of a *single sequence*. For example, if ρ violates Bell's inequalities then 'probability measures' μ_ρ do not admit the single-sequence foundation. This is a direct consequence of various 'ensemble derivations' of Bell's inequalities (see for example [6, 24, 25]).

Such extraordinary features are not in contradiction with physical experience (see [29]): after a specification of the measurement context, contextual statistics is reduced to classical statistics.

On the other hand, a more careful analysis shows [18] that an additional 'consistency condition' should be incorporated in the concept of subquantum reality. Roughly speaking, the condition forbids conceretizations of subquantum states of the system without context prespecifications (at least in situations where Bell's inequalities are violated).

(ii) As already mentioned, probability measures $\mu_\rho : \mathfrak{P} \to [0, 1]$ are generally not extendable (as probability measures) on the σ-field $B(\Omega)$. Moreover, certain correlated multi-particle states give *perfect correlations* which are not interpretable in terms of classical probability measures. This follows from the results of the work [20], in accordance with which there exist subquantum events $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \in \mathfrak{P}$ and correlated four-particle states ρ such that

$$\mu_\rho(\Lambda_i) = 1 \qquad \text{and} \qquad \Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \cup \Lambda_4 = \varnothing. \tag{4.1}$$

Clearly, such a possibility is forbidden in the framework of classical statistics. It is worth noting that the possibility of pathogological situations (4.1) does not lead to contradictions with physical experience, because events Λ_i figuring in (4.1) always belong to *different contexts.*

(iii) As an illustration of the formalism presented, we shall now examine an EPR-like situation with distant measurements on a two-particle system. We shall illustrate all important steps of the construction in this concrete example. A one-particle subquantum structure will be constructed first. The two-particle subquantum structure will then be obtained by simply taking the tensor product of the corresponding one-particle subquantum structures.

At the one-particle level, classical probability can be used in the statistical foundation of quantum states. In particular, subquantum structures can be treated within the framework of HV extensions [14].

Let us consider a one-particle quantum system. To simplify things, we shall assume that the Hilbert space H of quantum states of the particle is finite-dimensional. This corresponds, for example, to restriction on internal degrees of freedom (such as spin). The algebra Σ of relevant quantum observables then consists of all linear operators in H, that is, $\Sigma = L(H)$. Measurement contexts will be represented by maximal commutative *-subalgebras in Σ. Let $\mathbf{T}$ be the family of all contexts. Contexts $A \in \mathbf{T}$ are in a natural correspondence with orthogonal ray-projector decompositions $\hat{I} = \hat{P}_1 + \ldots + \hat{P}_k$, $k = \dim H$. The correspondence $\{\hat{P}_1, \ldots, \hat{P}_k\} \leftrightarrow A$ is given by the bicommutant. Operators $\{\hat{P}_1, \ldots, \hat{P}_k\}$ are characterized as minimal non-trivial projectors of A.

From the viewpoint of physical experience, the realization of a context $A \in \mathbf{T}$ is equivalent to the measurement of any observable $\hat{a} = \hat{a}^+ \in A$ which generates A. In this case $\hat{a} = \alpha_1 \hat{P}_1 + \ldots + \alpha_k \hat{P}_k$ where $\alpha_1, \ldots, \alpha_k$ are mutually different real numbers.

The elements of the spectrum $X(A)$ are naturally labelled by projectors $\hat{P}_1, \ldots, \hat{P}_k$. More precisely,

$$X(A) = \{\omega_{1,A}, \ldots, \omega_{k,A}\}$$

where characters $\omega_{i,A} : A \to C$ are fixed by $\omega_{i,A}(\hat{P}_j) = \delta_{ij}$.

Let us consider the space

$$\Pi = \prod_{A \in \mathbf{T}} X(A)$$

endowed with a product topology ($X(A)$ are endowed with discrete topology). We shall interpret the elements of Π as possible complete (subquantum) states of the particle. In other words, it is assumed that the particle is, in each moment of time, in some state $\omega \in \Pi$, unknown from the point of view of quantum description.

Further, it is assumed that each quantum observable $\hat{a} \in L(H)$, in each subquantum state $\omega \in \Pi$, possesses a definite value, if the measurement context $A \ni \hat{a}$ is specified. Let this value be defined by

$$F_A(\hat{a})(\omega) = \pi_A(\omega)(\hat{a})$$

where $\pi_A : \Pi \to X(A)$ is the Ath coordinate projection. The map $F_A : A \to C(\Pi)$ is a *-monomorphism.

Now, it is easy to see that for each quantum state $\psi \in H, |\psi| = 1$, there exists a probability measure μ_ψ on the Baire σ-field of Π satisfying

$$\langle\psi|\hat{a}|\psi\rangle = \int_\Omega F_A(\hat{a})(\omega)\, \mathrm{d}\mu_\psi(\omega) \qquad \text{for each } A \in \mathbf{T} \text{ and } \hat{a} \in A. \tag{4.2}$$

Indeed, it is sufficient to take the product of $X(A)$-measures $\mu_{\psi,A}$ where

$$\mu_{\psi,A}(\{\omega_{i,A}\}) = \langle\psi|\hat{P}_i|\psi\rangle \qquad i \in \{1, \ldots, k\}.$$

The formula (4.2) justfies the ignorance interpretation of quantum states.

The space Π, together with a family of maps $F_A : A \to C(\Omega)$, gives an example of a causal subquantum model for the quantum particle described by $H, \Sigma = L(H)$ and $\mathbf{T}$. The model is *contextual* in the sense that for each $\omega \in \Pi$ one can find contexts $A, B \in \mathbf{T}$ and a quantum observable $\hat{a} \in A \cap B$ such that $F_A(\hat{a})(\omega) \neq F_B(\hat{a})(\omega)$. As already mentioned, contextuality is a characteristic property of all consistent causal refinements of quantum mechanics (if it is assumed that all quantum observables are realizable subquantally).

The necessity of contextuality naturally leads to the idea that 'true physical quantities' are not quantum observables, but quantum observables completed by contexts. This idea lies in the origin of the formalism of contextual extensions, from the viewpoint of which the complete description should be based, instead of Σ, on a finer algebra Σ' which 'takes care' of contextuality in a proper way.

In the situation under consideration, the simplest possibility for Σ' is the direct sum

$$\Sigma' = C(\Pi) \oplus \Sigma.$$

Let us define the 'forgetting homomorphism' $\phi: \Sigma' \to \Sigma$ to be the projection on the second summand. Finally, let us define homomorphisms $\{\iota_A: A \to \Sigma'; A \in \mathbf{T}\}$ (contextual completions of quantum observables) by

$$\iota_A(\hat{a}) = F_A(\hat{a}) \oplus \hat{a}.$$

The triplet $(\Sigma', \phi, \{\iota_A; A \in \mathbf{T}\})$ is an HV extension (in the sense of [14]) of $(\Sigma, \mathbf{T})$. The elements of the space Π can be naturally viewed as characters of Σ'.

We pass to the consideration of a two-particle system. In order to simplify the construction, we shall assume that particles are distinguishable. Let the ith particle $(i \in \{1, 2\})$ be described, at a quantum level, by a Hilbert state-space H_i, the quantum algebra $\Sigma_i = L(H_i)$ and the family $\mathbf{T}_i$ of corresponding measurement contexts. The quantum description of the composite system is then based on the Hilbert space $H = H_1 \otimes H_2$ and the quantum algebra $\Sigma = L(H) = \Sigma_1 \otimes \Sigma_2$. Concerning measurement contexts, we should now distinguish simple (one-particle) and composite (two-particle) ones.

One-particle contexts are of the form $\tilde{A}_1 = A_1 \otimes id_2$ or $\tilde{A}_2 = id_1 \otimes A_2$, where $A_i \in \mathbf{T}_i$ and id_i is the identity in Σ_i. (In the following, it will be assumed that $\tilde{A}_i$ is identified with A_i, in a natural manner.) Two-particle contexts, corresponding to coincidence measurements, are then represented by sets of the form $A = \{A_1, A_2\}$ (with $\mathrm{dom}(A) = A_1 \otimes A_2$ and $A_i \in \mathbf{T}_i$). Let us denote by $\mathbf{T}$ ($\hat{\mathbf{T}}$) the set of all simple (simple and composite) contexts.

For the subquantum description of single particles, we shall use already constructed HV extensions $(\Sigma_i', \phi_i, \{\iota_{i,A}; A \in \mathbf{T}_i\})$. For the subquantum description of the composite system, it is natural to use the direct product of one-particle HV extensions. More precisely, let us consider the tensor product

$$\Sigma' = \Sigma_1' \otimes \Sigma_2'$$

and homomorphisms $\phi = \phi_1 \otimes \phi_2 : \Sigma' \to \Sigma$, $\iota_{A_1}: A_1 \to \Sigma'$ and $\iota_{A_2}: A_2 \to \Sigma'$.

Here $A_i \in \mathbf{T}_i$ and

$$\iota_{A_1}(\hat{a}_1 \otimes id_2) = \iota_{1,A_1}(\hat{a}_1) \otimes 1_2' \qquad \iota_{A_2}(id_1 \otimes \hat{a}_2) = 1_1' \otimes \iota_{2,A_2}(\hat{a}_2)$$

while $\hat{a}_i \in A_i$ and $1_i'$ is the unity of Σ_i'.

If $A = \{A_1, A_2\}$ then the formula

$$\iota_A(\hat{a}_1 \otimes \hat{a}_2) = \iota_{A_1}(\hat{a}_1)\iota_{A_2}(\hat{a}_2)$$

gives the unique *-homomorphism $\iota_A: \mathrm{dom}(A) \to \Sigma'$ which extends $\iota_{A_1}(\hat{a})$ and $\iota_{A_2}(\hat{a})$.

It is easy to see that the triplet $(\Sigma', \phi, \{\iota_A; A \in \mathbf{T}\})$ is an LCHV extension of $(\Sigma, \mathbf{T})$. This fact justifies the possibility of interpreting the algebra Σ' as a base for the complete description of the composite system. In the framework of such a description, possible states of the system are given by characters of Σ', which are represented by pairs (ω_1, ω_2) $(\leftrightarrow \omega_1 \otimes \omega_2)$ where $\omega_i \in \Pi_i$. Let $\Pi = \Pi_1 \times \Pi_2$. For each $a' \in \Sigma'$ and $\omega \in \Pi$ the number $\omega(a')$ can be interpreted as the value of a' in the state ω. In particular, the value $F_A(\hat{a})(\omega) = \omega(\iota_A(\hat{a}))$ of $\iota_A(\hat{a})$ in ω can be interpreted as the value of the quantum observable $\hat{a} \in \mathrm{dom}(A)$ relative to the context $A \in \hat{\mathbf{T}}$ in the subquantum state ω.

It is worth noting that the constructed structure is contextual, since contextuality is already present at the one-particle level.

However, only local contextuality enters the game because the value of a given one-particle observable in a given composite context depends only on the corresponding one-particle component of the latter. At the formal level, in the framework of identification $C(\Pi)=C(\Pi_1)\otimes C(\Pi_2)$, local contextuality is ensured by the following decomposability of corresponding *-monomorphisms $F_A:\mathrm{dom}(A)\to C(\Pi)$:

$$F_A(\hat{a}_1\otimes\hat{a}_2)=F_{A_1}(\hat{a}_1)\otimes F_{A_2}(\hat{a}_2) \qquad \text{where } A=\{A_1, A_2\}.$$

From the subquantum viewpoint, quantum states $\psi\in H=H_1\otimes H_2$ are interpretable as entities carrying information about the lack of knowledge of subquantum states $\omega\in\Pi$. Of course, for states violating Bell's inequalities, such an interpretation is not possible within the framework of classical statistics. On the other hand, the lack-of-knowledge interpretation is always possible within the framework of 'contextual statistics', the domain of which is restricted to the family $\mathfrak{P}$ of quantally actualizable subquantum events. The family $\mathfrak{P}$ is union of Boolean algebras $\mathfrak{P}_A$ where A is a composite context. Every $\mathfrak{P}_A$ consists precisely of subquantum events actualizable in the context A. Explicitly, atoms of $\mathfrak{P}_A$ are given by

$$\Lambda_{i,j,A}=(\pi_{A_1}\times\pi_{A_2})^{-1}(\omega_{i,A_1}\otimes\omega_{j,A_2}) \qquad \text{where } A=\{A_1, A_2\}.$$

The events $\Lambda_{i,j,A}$ are subquantum counterparts of quantum events $\hat{P}_i\otimes\hat{Q}_j$, where $\{\hat{P}_1,\ldots,\hat{P}_k\}$ and $\{\hat{Q}_1,\ldots,\hat{Q}_l\}$ $(k=\dim H_1, l=\dim H_2)$ and partitions of unity in H_1 and H_2 corresponding to A_1 and A_2 respectively. Each quantum state $\psi\in H$ gives rise to a 'probability measure' $\mu_\psi:\mathfrak{P}\to[0,1]$ on $(\Omega,\mathfrak{P})$ such that

$$\mu_\psi(\Lambda_{i,j,A})=\langle\psi,(\hat{P}_i\otimes\hat{Q}_j)\psi\rangle.$$

This formula gives a consistent contextual analogue of (4.2). It justifies the above-mentioned interpretation of quantum states.

The construction presented can easily be generalized to the systems with three, or more, quantum particles. In the case when particles are identical (bosons or fermions), a technical difficulty arises because the composite Hilbert space (and quantum algebra) can no longer be obtained by taking tensor products of corresponding one-particle objects. However, such a difficulty can be overcome by allowing all numbers of particles in the game and, consequently, by fixing the quantum description on the corresponding Fock space. Because orthogonal decompositions of the one-particle Hilbert space (which appear in the consideration of mutually independent measurements 'localized' in some regions of physical space) naturally induce tensor decompositions of the Fock space (which allows us to build an LCHV extension, along the lines of the construction presented, by taking the tensor product of HV extensions associated with corresponding 'localized' Fock spaces). A construction of the type presented can also be applied to infinitely extended lattice systems [18].

(iv) As already point out, the subquantum space Ω associated with an arbitrary LCHV extension $(\Sigma', \phi, \{\iota_A; A\in\mathbf{T}\})$ is compact, in the *-weak topology of $(\Sigma')^*$. This natural topology on Ω can be characterized as a minimal topology with respect to which all maps $F_A(\hat{a})$ (where $A\in\mathbf{T}$, $\hat{a}\in A$) are continuous. However, such a choice of topology implies, in the case when each context $A\in\mathbf{T}$ is generated by its projectors, that Ω is *extremally disconnected.* For example, if $\Sigma=L(H)$ and $\dim(H)<\infty$ then Ω is necessarily of the above-mentioned kind.

On the other hand, the necessity of an extremally disconnected topology can be avoided, by adopting a slight generalization of the formalism. For example, it is sufficient to allow situations in which Σ' is a von Neumann algebra and the space Ω

consists of characters of some C^*-subalgebra $\tilde{\Sigma}' \subseteq \Sigma'$ everywhere ultraweakly dense in Σ' such that the evaluation map $\tilde{\Sigma}' \to C(\Omega)$ is extendable to a von Neumann algebra homomorphism $\Sigma' \to L^\infty(\Omega, \mu)$, where μ is a measure on Ω and $L^\infty(\Omega, \mu)$ is the algebra of essentially bounded μ-measurable functions on Ω. (The algebra $\tilde{\Sigma}'$ need not be unital. Unitality of $\tilde{\Sigma}'$ ensures compactness of Ω, otherwise we can say only that Ω is locally compact.)

Such a modification of the formalism, conceptually inessential, has the consequence that maps $F_A(\hat{a})$ are no longer necessarily continuous (because $\iota_A(\hat{a})$ do not necessarily belong to $\tilde{\Sigma}'$). Further, $F_A(\hat{a})$ can be left undefined on some negligible sets.

As a result, 'nice' topological properties of Ω become possible. In particular, various concrete models having a smooth manifold as a subquantum space can be incorporated in this modified scheme.

It is also worth noting that, in certain situations, some additional parameters enter the context specification. For example, in the model presented by Bell in [5], contexts correspond to *ordered* partitions of unity.

Let us briefly examine HV extensions based on one-particle Bohm–de Broglie type models [2, 8]. The corresponding multiparticle structure can be constructed, as in the example considered in the previous remark, by taking the direct product of one-particle extensions.

In the framework of the Bohm–de Broglie theory, an actual physical situation is specified by a particle position, and a pilot-wave configuration; the latter corresponds to the quantum state of the system. Consequently,

$$\Omega = Q \times CP(H)$$

where Q is the configuration manifold of the particle, H the Hilbert state space and $CP(H)$ the corresponding complex projective space.

In the Bohm–de Broglie theory every quantum measurement is reducible to a measurement of the particle position. This means that, if a measurement context A is specified, every quantum observable $\hat{a} \in A$ is represented by a measurable function $f_A(\hat{a})$ on Q (so that $f_A : A \to L^\infty(Q)$ is a $*$-monomorphism).

A rough scenario of the A-measurement is this. The measurement begins with the particle in some position $q \in Q$, and with some initial point-wave configuration ψ. The evolution of the wave is determined by the Schrödinger equation (with Hamiltonian including interaction with the measurement apparatus). The evolution of the particle is determined by the corresponding (ψ-dependent) velocity flow. After a given time $\tau > 0$, a position measurement is made. If the particle position is $q(\tau)$, then, by definition, the result of the measurement of $\hat{a}$ in context A is

$$F_A(\hat{a})(q, \psi) = f_A(q(\tau)).$$

Starting from the space Ω, $*$-monomorphisms $F_A : A \to L^\infty(\Omega)$, the algebra $\Sigma = L(H)$ and the family T of contexts we can construct the corresponding extensions, as in the example previously considered. It is important to mention that the two-particle subquantum structure constructed in such a way differs essentially from the two-particle Bohm–de Broglie structure, which is highly non-local.

Although the Bohm–de Broglie model, at the one-particle level, fits into the approach of this work, the background philosophies of the two approaches are different. In the Bohm–de Broglie model, measurement results are 'created' in the process of interaction between object (particle + wave) and measurement apparatus. On the other hand, one of main ideas in the concept of HV and LCHV extension is that of 'pre-existing values' which are just 'read off' in measurements.

(v) The subquantum theory based on LCHV extensions is causal, in the sense that quantum observables possess definite values in subquantum states. However, we said nothing about *dynamics*. In fact, two general possibilities are left open.

The first possibility is in a causal dynamics. From the algebraic viewpoint, it is natural to introduce such a dynamics via a one-parameter group of automorphisms of $\tilde{\Sigma}'$, which then induces the evolution on Ω and which is projectable, via $\phi:\Sigma'\to\Sigma$, to the quantum evolution.

The second possibility lies in an inherently stochastic dynamics. Nelson's stochastic mechanics [11, 26] provide an important example of a subquantum structure of this kind: the particle evolution is assumed to be stochastic, described by an appropriate diffusion process in the configuration space Q.

(vi) A large class of LCHV-type extensions can be obtained [18] by reinterpreting deformation quantization constructions [1, 7, 31] in terms of hidden variables.

A general scheme is this. One starts with a symplectic (or Poisson) manifold Ω and introduces an associative *-algebra structure on the space $\mathscr{A}_h$ of formal h-power series with coefficients belonging to $C^\infty(\Omega)$. For this product, it is required that replacing $h\to 0$ gives a homomorphism $\mathrm{cl}:\mathscr{A}_h\to C^\infty(\Omega)$ such that (correspondence principle)

$$\mathrm{cl}((i/h)[a,b])=\{\mathrm{cl}(a),\mathrm{cl}(b)\} \tag{4.3}$$

where $\{,\}$ is the Poisson bracket on Ω. The usual quantum formalism arises by taking a representation D (in which h becomes a positive number) of an appropriate (sufficiently rich) *-subalgebra $\tilde{\mathscr{A}}_h$ of $\mathscr{A}_h$ in a Hilbert space H. The elements of the uniform closure Σ of $D(\tilde{\mathscr{A}}_h)$ play the role of 'quantum observables'.

Such a scheme can be linked with our subquantum one in the following way. The space Ω plays the role of the subquantum space, the algebra $\tilde{\mathscr{A}}_h$, completed in an appropriate way, becomes the C^*-algebra of 'subquantum observables' and D extends to the 'forgetting homomorphism' $\phi:\Sigma'\to\Sigma$.

In order to illustrate this, let us examine a quantization of the algebra of functions of the $2n$-dimensional torus

$$T=(S^1)^n\times(S^1)^n \qquad \text{where } S^1=\{z\in C;\,|z|=1\}.$$

Let $u:S^1\to C$ be the inclusion map and, for $k\in\{1,\dots,n\}$, let u_k, v_k be the compositions $u\pi_k$, $u\pi_{k+n}$ where π are coordinate projections. We shall assume that T is endowed with the following symplectic structure:

$$\vartheta=\sum_{k=1}^{n} d(\arg(u_k))\wedge d(\arg(v_k)).$$

Let the space $\mathscr{A}_h$ be endowed with the product $m:\mathscr{A}_h\otimes\mathscr{A}_h\to\mathscr{A}_h$ given by (see [1])

$$m=m_0\exp\left((h/2\mathrm{i})\left(\sum_{k=1}^{n} X_k\otimes Y_k-Y_k\otimes X_k\right)\right)$$

where m_0 is the product induced from $C^\infty(T)$ and X_k, Y_k are canonical basis vector fields (dual to $d(\arg(u_k)), d(\arg(v_k))$). The space $\mathscr{A}_h$, with the product m, and *-involution induced from $C^\infty(T)$ ($h^*=h$) becomes a *-algebra. It is easy to see that (4.3) holds.

Let U_k, V_k be unitary elements in $\mathscr{A}_h$ which correspond to u_k, v_k respectively and let $\tilde{\mathscr{A}}_h\subseteq\mathscr{A}_h$ be a *-subalgebra generated by elements U_k, V_k and $\mathrm{e}^{\mathrm{i}h/z}$. For each $\alpha>0$

and $\lambda=(\lambda^1,\ldots,\lambda^n)\in R^n$ we shall denote by $D_{\lambda,\alpha}$ a representation of $\tilde{\mathscr{A}}_h$ in a Hilbert space $H=L^2((S^1)^n)$ defined by

$$D_{\lambda,\alpha}(\mathrm{e}^{\mathrm{i}h/2})=\mathrm{e}^{\mathrm{i}\alpha/2}\hat{I} \qquad D_{\lambda,\alpha}(U_k)\psi=u_k\psi$$

$$[D_{\lambda,\alpha}(V_k)\psi](z_1,\ldots,z_n)=\mathrm{e}^{\mathrm{i}\lambda^k}\psi(z_1,\ldots,\mathrm{e}^{-\mathrm{i}\alpha}z_k,\ldots,z_n).$$

Using the family $\{D_{\lambda,\alpha};\,\lambda\in R^n,\,\alpha>0\}$ we can introduce a *-norm on $\tilde{\mathscr{A}}_h$, by requiring

$$|f|=\sup_{\alpha,\lambda}|D_{\lambda,\alpha}(f)|.$$

We shall denote by Σ' the completion of $\tilde{\mathscr{A}}_h$ with respect to this norm.

The map $\mathrm{cl}:\tilde{\mathscr{A}}_h\to C^\infty(T)$ is uniquely extendable to a surjective *-homomorphism $\mathrm{cl}:\Sigma'\to C(T)$. The elements of T are interpretable as characters of Σ'. All representations $D_{\lambda,\alpha}$ are uniquely extendable from $\tilde{\mathscr{A}}_h$ to Σ'. Let us fix λ and α such that $\mathrm{e}^{\mathrm{i}\alpha}$ is not a root of unity, and consider a representation $D_{\lambda,\alpha}$ as a *-homomorphism $\phi:\Sigma'\to\Sigma$. Here, $\Sigma=D_{\lambda,\alpha}(\Sigma')$ is a C^* subalgebra in $L(H)$, interpretable as consisting of quantum observables. It is worth noting that Σ is essentially the algebra of 'functions' on the quantum tori [12].

We pass on to the description of 'measurement contexts'. For each $k\in\{1,\ldots,n\}$ and elementary vector $(p,q)\in Z^2\backslash\{(0,0)\}$ we shall denote by $A^k_{pq}\subseteq\Sigma$ the C^*-subalgebra generated by $\phi(U^p_kV^q_k)$. It is easy to see that the spectrum $\sigma[\phi(U^p_kV^q_k)]$ (coinciding with the spectrum $X(A^k_{pq})$ of A^k_{pq}) coincides with S^1. Let $\mathbf{T}$ be the set of all such subalgebras. For $A=A^k_{pq}$, let $\iota_A:A\to\Sigma'$ be a *-homomorphism determined by $\iota_A(\phi(U^p_kV^q_k))=\mathrm{e}^{\mathrm{i}(\alpha-h)pq/2}U^p_kV^q_k$. Concerning composite contexts, let them be defined as sets of the form

$$A=\{A^{r_1}_{p_1q_1},\ldots,A^{r_j}_{p_jq_j}\} \qquad \text{where } 1\leqslant r_1<\ldots<r_j\leqslant n.$$

Then, the triplet $(\Sigma',\phi,\{\iota_A;\,A\in\mathbf{T}\})$ is an LCHV extension of $(\Sigma,\mathbf{T})$.

(vii) In various relevant situations, the system considered possesses, at the quantum level, some symmetry. It is interesting to ask whether this symmetry is preserved during the passage from the quantum to the subquantum level of description.

Let G be a group of automorphisms of Σ such that families $\mathbf{T}$ and $\hat{\mathbf{T}}$ are G-invariant. By definition, an LCHV extension $(\Sigma',\phi,\{\iota_A;\,A\in\mathbf{T}\})$ of $(\Sigma,\mathbf{T})$ is *G-covariant* iff there exists an action $\alpha_G:G\to\mathrm{Aut}(\Sigma')$ of G by automorphisms of Σ' such that

$$\iota_{g(A)}gi_A=\alpha_G(g)\iota_A \qquad \text{for each } A\in\mathbf{T} \text{ and } g\in G. \tag{4.4}$$

Of course, in a general case the extension $(\Sigma',\phi,\{\iota_A;\,A\in\mathbf{T}\})$ will not be G-covariant. But if it is, then, owing to (4.4) and (2.1; iv), the action α_G is unique. It is easy to see that $(Lc(\Sigma,\mathbf{T}),\hat{\phi},\{\hat{\iota}_A;\,A\in\mathbf{T}\})$ is always G-covariant.

If $(\Sigma',\phi,\{\iota_A;\,A\in\mathbf{T}\})$ is G-covariant, then the action α_G naturally induces an action d_G of G by automorphisms of the subquantum space Ω $(d_G(g):\omega\to\omega\alpha_G(g^{-1}))$. All canonically associated subquantum entities are G-covariant in a natural manner. If, in addition, a representation D of Σ in a Hilbert space H is given, together with an unitary representation U of G in the same space, such that

$$Dg=U(g)D(\ )U^+(g) \qquad \text{for each } g\in G$$

then the 'interpreter maps' figuring in proposition 3.3 are also G-covariant, in a natural manner.

(viii) There exists a certain similarity between *classical mechanics* and subquantum mechanics considered in this study. The similarity may be twofold: as first, both structures include the concept of complete states, in which all physical quantities describing the system have definite values. Thus, an analogy between the subquantum space Ω and the phase space in classical mechanics, naturally emerges.

Secondly, we can speak about dynamical similarity between classical and subquantum mechanics. Such a similarity is realized in those subquantum theories in which subquantum space has a symplectic manifold structure, and dynamics is of the Hamiltonian form [16, 18].

In the framework of the above-mentioned subquantum–classical analogies, quantum theory appears (because of the ignorance interpretation of quantum states) as something like classical statistical mechanics on the subquantum space.

However, in contrast to the relation

classical mechanics $\leftrightarrow$ classical statistical mechanics

the relation

subquantum mechanics $\leftrightarrow$ quantum mechanics

shows contextual features.

Another example of a phenomenon not appearing in the classical world is existence of properties of the system which are not *actualizable jointly*. This can be understood as a manifestation of *complementarity*. At a formal level, complementarity is reflected in the non-commutativity of the algebra of observables. Since complementarity is a fundamental characteristic of the quantum world, every 'deeper' theory extending quantum mechanics should, in some sense, preserve it. From the viewpoint of this study, complete description of physical reality should be based on appropriate (necessarily non-commutative) extensions of quantum algebras.

Approximativity of quantum description is reflected in the fact that the quantum algebra Σ can be obtained from the subquantum algebra Σ' by factorizing through the 'ideal of hidden variables' $\mathrm{hv}(\Sigma') = \ker(\phi)$. Clearly, the replacement $\Sigma \to \Sigma'$ preserves the information about complementary observables.

Not all extensions Σ' of the quantum algebra Σ are subquantally relevant: among other properties, the algebra Σ': should possess sufficiently many characters, because they form the subquantum space Ω.

The description of the system in terms of Ω is understandable as a 'classical type' approximation of the complete description. The passage from the Σ'-description to the classical Ω-description is obtained by factorizing through the ideal $J = \mathrm{com}(\Sigma')$.

The quantum algebra Σ is not canonically determined by Σ' (in contrast to Ω and J). However, in the case when the algebra Σ is 'purely quantum', in the sense of not admitting dispersion-free states, all the information about Σ is contained in J, because in this case $\phi(J) = \mathrm{com}(\Sigma) = \Sigma$.

Let us assume that Ω and J are given. An important problem naturally arises: how do we compute all possible algebras Σ' having Ω as a subquantum space and satisfying $J = \mathrm{com}(\Sigma')$?

It is also of interest, if Σ' is given, to investigate all possible 'quantum approximations' Σ (equivalently, all possible ideal $\mathrm{hv}(\Sigma')$).

From the mathematical viewpoint, a framework for problems of the above-mentioned kind is, essentially, the theory of Brown *et al* [10]. This connection opens of the possibility of applying non-commutative-geometric [12] methods in subquantum mechanics.

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